TRANSFER FUNCTIONS FOR THE TEMPERATURE OF A BODY IN THE PRESENCE OF GENERALIZED THERMAL EFFECTS

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The transfer functions and approximate differential equations of the interrelation between the temperature of a body and generalized thermal effects are obtained. The coefficients of the transfer functions are found with consideration of the shape of the investigated body.

The temperature in various objects is calculated on the basis of solving the equation of heat conductivity with consideration of the geometric and thermophysical properties of the investigated body, conditions of its heat exchange with the ambient medium, and character of the temporal change of the regime energy factors. The complete solution of the problem of heat transfer, even with a number of simplifying premises, cannot always be obtained in a form convenient for engineering use. The indicated difficulties can be eliminated to some extent by constructing combined solutions suitable for bodies of different shapes and limiting the problem to the search for the characteristic temperatures, for example, central, average surface, and average volume temperatures.

The body under consideration belongs to a class of convex polyhedra, and the thermophysical properties do not change with time and do not depend on temperature. The thermal regime of the body is determined by the following thermal effects:

1) temperature of the external medium $t(\tau)$;

2) external energy sources whose density per unit surface of the body is equal to $q(\tau)$, W/m^2;

3) internal uniformly distributed energy sources (sinks) whose density per unit volume of the body is $w(\tau)$, W/m^3;

4) temperature of the medium $v(\tau)$ passing through the body.

The presence of the fourth regime factor means that the body is some ordered structure with open interconnecting pores. The intensity of heat transfer of the body with the external medium is characterized by the heat-transfer coefficient α, $W/m^2 \cdot deg$, which retains a constant value for all portions of the body's surface. The intensity of heat transfer with the internal medium is determined by the parameter b, $W/m^3 \cdot deg$, which also remains constant for all elements of the body's volume.

The integral geometric properties of the body are given by its total volume V and external surface S.

With these premises the problem of heat conductivity is linear. To simplify its formulation and solution we will henceforth use temperature $u(r, \tau)$, which is the average temperature on some surface σ located within the body. The interrelation between the averaging surface and the space coordinate r is governed by the equation

$$\sigma = S\left(\frac{r}{R}\right)^n = S\rho^n. \tag{1}$$

It is necessary to note that coordinate r is not directly related with the usual (e.g., Cartesian) coordinates x, y, z; it is expedient to locate its origin at such a point within the body where the temperature gradient is equal to zero. For bodies of a regular shape having axes or planes of symmetry, the location

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of the point of the extreme value of the temperature usually coincides with the center of gravity of the body.

For a zero value of r the temperature-averaging surface contracts to a point, and for its maximum value, equal to the characteristic size of the body, σ becomes equal to the surface of the body, i.e.,

$$\sigma|_{r=0} = 0, \qquad \sigma|_{r=R} = S.$$

The volume element of the body dV is found according to the relation $dV = \sigma dr$. Problems of determining n and R for different bodies are discussed in [1]. The interrelation between the geometric quantities is expressed by the relation

$$\frac{V}{S} = \frac{R}{n+1}. \tag{2}$$

On the basis of the definitions given above we can obtain the following one-dimensional equation of heat conductivity for a homogeneous isotropic body:

$$\frac{\partial u}{\partial \tau} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{n}{r}\frac{\partial u}{\partial r} - \frac{b}{\lambda}u\right) + \frac{1}{c\gamma}w + \frac{b}{c\gamma}v \tag{3}$$

with boundary conditions

$$\left(\lambda\frac{\partial u}{\partial r} + \alpha u\right)\Big|_{r=R} = q + \alpha t, \quad \frac{\partial u}{\partial r}\Big|_{r=0} = 0 \tag{4}$$

and initial condition

$$u(r, \tau)\Big|_{\tau=0} = 0. \tag{5}$$

Applying to (3), (4), and (5) the integral Laplace transform [2], we can obtain a general relation associating the temperature transform with thermal effects:

$$U(\rho, s) = Y_e(\rho, s)Z_e(s) + Y_i(\rho, s)Z_i(s). \tag{6}$$

Here

$$\left.\begin{aligned} Z_e(s) &= T(s) + \frac{1}{\alpha}Q(s), \\ Z_i(s) &= V(s) + \frac{1}{b}W(s) \end{aligned}\right\} \tag{7}$$

are transforms of the variables

$$z_e(\tau) = t(\tau) + \frac{1}{\alpha}q(\tau), \quad z_i(\tau) = v(\tau) + \frac{1}{b}w(\tau). \tag{8}$$

Function $z_e(\tau)$ and its Laplace transform $Z_e(s)$, which determine heat transfer of the body in the case of the total effect of the temperature of the external medium and energy sources on the surface, will be called the generalized external heat effect. Function $z_i(\tau)$ and its transform $Z_i(s)$, which reflect the total effect of the internal energy factors, can be called by analogy the generalized internal heat effect.

The transforms of the effects are related with the transform of the temperature of the body by the functions $Y_e(\rho, s)$ and $Y_i(\rho, s)$, which will be called the transfer functions for the temperature of the body with respect to the external and internal heat effects. A noteworthy characteristic of these functions is that they are determined only by the intrinsic parameters of the body (its dimensions, thermophysical properties, and heat-transfer coefficient) and do not depend on the magnitude and law of variation of the regime factors. The investigated body can be regarded as a heat system transforming the input effects $z_e(\tau)$ and $z_i(\tau)$ into the output quantity – the temperature of the body $u(\rho, \tau)$.

On the basis of relation (6) we can obtain the equation for determining the transforms of the central, average surface and average volume temperatures of the body:

$$\left.\begin{aligned} U_0(s) &= Y_{e,0}(s)Z_e(s) + Y_{i,0}(s)Z_i(s), \\ U_S(s) &= Y_{e,S}(s)Z_e(s) + Y_{i,S}(s)Z_i(s), \\ U_V(s) &= Y_{e,V}(s)Z_e(s) + Y_{i,V}(s)Z_i(s), \end{aligned}\right\} \tag{9}$$

which are found from (6) respectively for $\rho = 0$, $\rho = 1$, and as a result of integration over the volume of the body with consideration of relations (1) and (2).

The structure of the transfer functions is shown in Table 1, where the following notations are introduced:

$$\beta = \mu R, \qquad \mu = \sqrt{\frac{b}{\lambda} + \frac{s}{a}}; \tag{10}$$

$$\nu = \frac{n-1}{2}, \qquad \zeta = \frac{\alpha R}{\lambda}; \tag{11}$$

$$G(s) = \beta^{-\nu}\,[\zeta\, I_\nu(\beta) + \beta\, I_{\nu+1}(\beta)]. \tag{12}$$

If internal "convective" energy sinks are absent in the body, i.e., b = 0, formulas (10) are simplified:

$$\beta = \mu R = \sqrt{\frac{s}{a}}\,R, \tag{13}$$

and the heat effect $z_i(\tau)$ and its transform $Z_i(s)$ are determined by the relations:

$$z_i(\tau) = \frac{R^2}{\lambda}\,w(\tau), \qquad Z_i(s) = \frac{R^2}{\lambda}\,W(s). \tag{14}$$

The form of notation for transfer function Y_e and its particular cases with consideration of (13) remains as before (Table 1), and function Y_i is found from the equation

$$Y_i(\rho, s) = \frac{1}{\beta^2}\left[1 - Y_e(\rho, s)\right] = \frac{1}{\beta^2}\left[1 - \rho^{-\nu}\,\frac{\zeta\, I_\nu(\beta)}{\zeta\, I_\nu(\beta) + \beta\, I_{\nu+1}(\beta)}\right]. \tag{15}$$

For values of the index $\nu = -1/2$, 0, and 1/2, i.e., values of form factor n = 0, 1, and 2, we obtain from expressions (6) and (9) rigorous solutions of the problems of heat conductivity for bodies of a canonical shape (plate, cylinder, sphere). Transfer functions Y_e and Y_i accordingly change to exact transfer functions for a plate, cylinder, and sphere, which are given in [3] or can be found from solutions in [2]. For other values of n Eq. (3) and solutions (6) and (9) are approximate and for the majority of practical problems provide an accuracy sufficient for engineering calculations [3].

The formal change from solution (6) to the true values of the temperature $u(\rho, \tau)$ in the presence of continuously varying heat effects $z_e(\tau)$ and $z_i(\tau)$ can be done on the basis of Duhamel's theorem or theorem of convolution of functions [2]

$$u(\rho, \tau) = \int_0^\tau y_e(\rho, \theta)\, z_e(\tau - \theta)\, d\theta + \int_0^\tau y_i(\rho, \theta)\, z_i(\tau - \theta)\, d\theta. \tag{16}$$

We note that in the theory of linear dynamic systems the functions y_e and y_i are usually called impulse or weighting functions of the system, which corresponds to a special case of effects given in the form of unit impulse functions or Dirac delta functions.

If the generalized heat effects vary in time, then despite the use of the one-dimensional equation of heat conductivity (3) the change from Eqs. (6) and (9) to the actual temperatures is rather complicated and the solutions obtained are not always convenient for practical use. One of the possible ways of simplifying the problem is to replace the transfer functions (Table 1) by approximate expressions which should be simple in form, permit changing back to the original, mainly by means of tables of operational correspondences, and take into account the most essential properties of the initial ("exact") transfer functions. It is reasonable to represent Y_e and Y_i in the form of the ratio of two functions $\Phi(\rho, s)$ and G(s), each of which does not have singular points (poles) and can be expanded in a power series with respect to parameter s [2, 3]:

$$Y_e(\rho, s) = \frac{\Phi_e(\rho, s)}{G(s)}, \qquad Y_i(\rho, s) = \frac{\Phi_i(\rho, s)}{G(s)}. \tag{17}$$

If in the expansion of the numerator and denominator of (17) we restrict ourselves to several terms, in place of the initial functions Y_e and Y_i we obtain their approximate expressions, which are the ratio of two power polynomials, i.e., ordinary rational functions of parameter s.

For the important particular case b = 0 (absence of "convective" energy sinks within the body), expanding functions (17) figuring in Eq. (6) and restricting ourselves to m terms in the numerator and n terms in the denominator, we obtain:

TABLE 1. Transfer Functions for Temperature of Body

Temperature	Function Y_e	Function Y_i
Local $u(r, \tau)$	$Y_e(\rho, s) = \frac{\zeta(\beta\rho)^{-\nu} I_\nu(\beta\rho)}{G(s)}$	$Y_i(\rho, s) = \frac{b}{\lambda\mu^2}[1-Y_e(\rho, s)]$
Central $u_0(\tau)$	$Y_{e,0}(s) = \frac{\frac{\zeta}{2^\nu \Gamma(1+\nu)}}{G(s)}$	$Y_{i,0}(s) = \frac{b}{\lambda\mu^2}[1-Y_{e,0}(s)]$
Average surface $u_S(\tau)$	$Y_{e,S}(s) = \frac{\zeta\beta^{-\nu} I_\nu(\beta)}{G(s)}$	$Y_{i,S}(s) = \frac{b}{\lambda\mu^2}[1-Y_{e,S}(s)]$
Average volume $u_V(\tau)$	$Y_{e,V}(s) = \frac{2(\nu+1)\zeta\beta^{-(\nu+1)} I_{\nu+1}(\beta)}{G(s)}$	$Y_{i,V}(s) = \frac{b}{\lambda\mu^2}[1-Y_{e,V}(s)]$

TABLE 2. Coefficients of Transfer Functions

Coefficients	Form factor of body	
	ν	n
a_1	$\frac{R^2}{2(1+\nu)a} \frac{1+\frac{1}{2}\zeta}{\zeta}$	$\frac{R^2}{(1+n)a} \frac{1+\frac{1}{2}\zeta}{\zeta}$
a_2	$\frac{R^4}{8(1+\nu)(2+\nu)a^2} \frac{1+\frac{1}{4}\zeta}{\zeta}$	$\frac{R^4}{2(1+n)(3+n)a^2} \frac{1+\frac{1}{4}\zeta}{\zeta}$
b_1	$\frac{R^2\rho^2}{4(1+\nu)a}$	$\frac{R^2\rho^2}{2(1+n)a}$
$b_{1,S}$	$\frac{R^2}{4(1+\nu)a}$	$\frac{R^2}{2(1+n)a}$
$b_{1,V}$	$\frac{R^2}{4(2+\nu)a}$	$\frac{R^2}{2(3+n)a}$
b_2	$\frac{R^4\rho^4}{32(1+\nu)(2+\nu)a^2}$	$\frac{R^4\rho^4}{8(1+n)(3+n)a^2}$
$b_{2,S}$	$\frac{R^4}{32(1+\nu)(2+\nu)a^2}$	$\frac{R^4}{8(1+n)(3+n)a^2}$
$b_{2,V}$	$\frac{R^4}{32(2+\nu)(3+\nu)a^2}$	$\frac{R^4}{8(3+n)(5+n)a^2}$

$$Y_e(\rho, s) = \frac{1+\sum_{k=1}^{m} b_k s^k}{1+\sum_{k=1}^{n} a_k s^k}, \tag{18}$$

$$Y_i(\rho, s) = \frac{a}{R^2} \frac{\sum_{k=1}^{m}(a_k - b_k) s^{k-1}}{1+\sum_{k=1}^{n} a_k s^k}. \tag{19}$$

In expressions (18), (19) summation is carried out over numbers k = 1, 2, . . . , m, . . . , n, where m ≤ n.

The coefficients of the expansion of transfer functions (17) for the local temperature of the body $u(\rho, \tau)$ are found by the formulas:

$$a_k = \frac{\Gamma(\nu+1)}{2^{2k-1}\Gamma(k)\Gamma(k+\nu+1)} \frac{1+\frac{\zeta}{2k}}{\zeta} \left(\frac{R^2}{a}\right)^k, \tag{20}$$

$$b_k = \frac{\Gamma(\nu+1)}{2^{2k} k\Gamma(k)\Gamma(k+\nu+1)} \left(\frac{R^2\rho^2}{a}\right)^k, \tag{21}$$

$$a_k - b_k = \frac{\Gamma(\nu+1)}{2^{2k-1}\Gamma(k)\Gamma(k+\nu+1)} \frac{1+\frac{\zeta}{2k}(1-\rho^{2k})}{\zeta}\left(\frac{R^2}{a}\right)^k. \quad (22)$$

The replacement of the transfer functions (Table 1) by their approximate expressions (18) and (19) permits changing from partial differential equations (3) to ordinary differential equations. Let the initial values of $u(\rho, \tau)$, $z_e(\tau)$, and $z_i(\tau)$ together with their derivatives with respect to time up to orders m and n inclusively be equal to zero. Then as a result of inverse transformation of Eq. (6) with consideration of (18) and (19) we obtain

$$u(\rho,\tau) + \sum_{k=1}^{n} a_k \frac{d^k u(\rho,\tau)}{d\tau^k} = z_e(\tau) + \sum_{k=1}^{m} b_k \frac{d^k z_e(\tau)}{d\tau^k} + \frac{a}{R^2}(a_1 - b_1) z_i(\tau) + \frac{a}{R^2}\sum_{b=2}^{m}(a_k - b_h)\frac{d^{k-1} z_i(\tau)}{d\tau^{k-1}}, \quad (23)$$

where

$$z_e(\tau) = t(\tau) + \frac{1}{\alpha} q(\tau); \quad z_i(\tau) = \frac{R^2}{\lambda} w(\tau). \quad (24)$$

Similar equations can be obtained from expressions (9) for the characteristic temperatures of the body $u_0(\tau)$, $u_S(\tau)$, and $u_V(\tau)$.

The number of terms of the expansion retained can vary depending on the requirements of practical accuracy. However, for many applied problems good results are obtained already with the use of the second approximation, when m = 1 and n = 2. In this approximation the transfer functions for the temperature have the form

$$Y_e(\rho, s) = \frac{1 + b_1 s}{1 + a_1 s + a_2 s^2}, \quad Y_i(\rho, s) = \frac{a}{R^2}\frac{a_1 - b_1 + (a_2 - b_2) s}{1 + a_1 s + a_2 s^2}; \quad (25)$$

accordingly Eq. (23) is simplified:

$$u(\rho,\tau) + a_1 \frac{du(\rho,\tau)}{d\tau} + a_2 \frac{d^2 u(\rho,\tau)}{d\tau^2} = z_e(\tau) + b_1 \frac{dz_e(\tau)}{d\tau} + \frac{a}{R^2}(a_1 - b_1) z_i(\tau) + \frac{a}{R^2}(a_2 - b_2)\frac{dz_i(\tau)}{d\tau}. \quad (26)$$

Unlike b_k coefficients a_k do not depend on coordinate ρ. The values of the coefficients are given in Table 2.

To determine the characteristic temperature of the body the function $u(\rho, \tau)$ in Eqs. (23) and (26) is replaced accordingly by $u_0(\tau)$, $u_S(\tau)$, or $u_V(\tau)$, and coefficients b_1 and b_2 in expressions (23), (25), and (26) are replaced by $b_{1,S}$ and $b_{1,V}$ (for the central temperature $b_{1,0} = b_{2,0} = 0$).

Transfer functions (18), (19), and (25) and the differential equations of the relation between the effects of the form (23) and (26) can be used for solving diverse problems of heat transfer.

NOTATION

γ	is the density;
c	is the specific heat;
λ, a	are the heat conductivity and thermal diffusivity of the body;
τ	is the time;
$\rho = r/R$	is the relative coordinate;
R	is the characteristic dimension;
n	is the form factor of the body;
s	is the Laplace transform parameter;
$u(\rho, \tau)$, $U(\rho, s)$	are the temperature of body and its Laplace transform;
$u_0(\tau)$, $u_S(\tau)$, $u_V(\tau)$	are the central, average surface, and average volume temperature of body;
$U_0(s)$, $U_S(s)$, $U_V(s)$	are the transforms of these temperatures;
Y_e, Y_i	are the transfer functions;
y_e, y_i	are the originals of transfer functions;
ζ	is the Biot number;
θ	is the variable of integration;
I_ν	is the cylindrical function of the second kind with real index ν;
Γ	is the gamma function.

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